

# Mathematical Appendix for “Underfunding in Terrorist Organizations”

Jacob N. Shapiro\*

David A. Siegel†

## 1 Appendix

This appendix contains proofs of all propositions in the text. The first three results utilize the techniques of monotone comparative statics.<sup>1</sup>

**Proof of Lemma 1:** In the general case, the first-order condition that implicitly specifies  $x^*$  is  $\frac{\partial C}{\partial x} = 0$ , which is

$$\frac{(\gamma v' - (1 - \gamma)p')(1 - \delta_M(pq_S + (1 - p)q_F)) - \delta_M(p'q_S - p'q_F)(\gamma v + (1 - \gamma)p)}{(1 - \delta_M(pq_S + (1 - p)q_F))^2} = 0. \quad (11)$$

The cross-partial of M’s objective with respect to  $x$  and  $q_S$  is

$$\frac{-\delta_M \gamma (pv' + vp')}{(1 - \delta_M(pq_S + (1 - p)q_F))^2} + \frac{2\delta_M p \frac{\partial C}{\partial x}}{(1 - \delta_M(pq_S + (1 - p)q_F))}. \quad (12)$$

The first term is strictly negative by assumption, while the second is zero by (11). Thus,  $x^*$  is decreasing in  $q_S$ . Since there is no cost to B of increasing  $q_S$  and a positive benefit to decreasing  $x$ , B maximizes his utility with  $q_S^* = 1$ . QED.

**Proof of Lemma 2:** We follow the same logic as in Lemma 1. The cross-partial of M’s objective with respect to  $x$  and  $q_F$  is

$$\frac{\delta_M(\gamma(-v'(1 - p) + p'v) + (1 - \gamma)p')}{(1 - \delta_M(pq_S + (1 - p)q_F))^2} + \frac{2\delta_M(1 - p)\frac{\partial C}{\partial x}}{(1 - \delta_M(pq_S + (1 - p)q_F))}. \quad (13)$$

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\*Princeton University.

†Florida State University.

<sup>1</sup>Specifically, Theorem 3 in Ashworth and Bueno de Mesquita (2006).

The second term is zero by (11). To discern the sign of the first term, we first solve for  $v'$  in (11), after having set  $q_s = 1$ , and then substitute this into (13). Doing so, after some algebra, makes clear that the first term is positive. Thus,  $x^*$  is increasing in  $q_F$ . Since there is no cost to B of increasing  $q_F$  and a positive benefit to decreasing  $x$ , B maximizes his utility with  $q_F^* = 0$ . QED.

**Proof of Proposition 1:** Assume that  $p$  is increasing and concave over the region in which an interior solution obtains, and that  $v(x)$  is increasing. Taking into account Lemmas 1 and 2,  $\frac{\partial C}{\partial x} = 0$  becomes

$$\frac{(\gamma v' - (1 - \gamma)p')(1 - \delta_M p) - \delta_M p'(\gamma v + (1 - \gamma)p)}{(1 - \delta_M p)^2} = 0. \quad (14)$$

The cross partial of this with respect to  $w$  is

$$\frac{-\gamma \delta_M v' p' - (1 - \gamma + \gamma \delta_M v) p''}{(1 - \delta_M p)^2} + \frac{2 \delta_M p' \frac{\partial C}{\partial x}}{1 - \delta_M p}. \quad (15)$$

The second term is zero by (14). The first is positive—implying that  $x^*$  is increasing in  $w_0$ —whenever

$$\gamma(v' p' + v p'') \delta_M < -(1 - \gamma) p''.$$

The term on the right hand side of this inequality is always positive by assumption. If the term on the left hand side is negative, then the inequality is true for all  $\delta_M$ . If it is positive, then by continuity there must exist a maximal  $\epsilon > 0$  such that the inequality holds whenever  $\delta_M \leq \epsilon$ . QED.

The remaining results for the “skimming” region utilize total differentiation of (4) and (7), with the appropriate functional forms inserted. Simplified, (4) and (7) become

$$\frac{1 - \delta_M p}{\beta p(1 - p)} - x^* \delta_M - \frac{1 - \gamma}{\gamma b} = 0, \quad (16)$$

$$\frac{\delta_M \beta p^2 (1 - p)^2}{(1 - \delta_M p)(2p - 1)} - c = 0. \quad (17)$$

The total derivative for  $w$  is computed directly from (17); that for  $x$  is computed using this result, along with the partial derivative of (16) and the fact that  $\frac{dx^*}{d\nu} = \frac{\partial x^*}{\partial \nu} + \frac{\partial x^*}{\partial w_0^*} \frac{dw_0^*}{d\nu}$ , where  $\nu$  is any parameter. Subtracting the latter from the former dictates how  $p$  changes, since it is increasing in the difference  $w_0^* - x^*$ . For the “transition” region we directly differentiate (16) with  $x^* = 0$ . For the “honest” region we directly differentiate (9).

Determining the effects of parameter variation on the cutoffs  $\gamma_0$  and  $\gamma_1$  is somewhat more complex, due to the fact that they each depend upon the equilibrium value of  $p$ , and the derivatives

of  $p$  with respect to the parameters are discontinuous at the cutoffs in many cases. Thus, to sign these comparative statics we look instead at relative changes in  $p$  between regions, relying upon the definition of the “transition” region as that in which  $p < p_0$  and  $x^* = 0$ . If  $p$  is decreasing faster in some parameter in the honest region than in the transition region, then the level of greed,  $\gamma_0$ , required for M to be willing to skim, and hence for B to have to take this potential into account, will increase. Otherwise it decreases. Likewise, if  $p$  is decreasing faster in the skimming region than in the transition region, the level of greed for which choosing  $w_0$  such that  $x^* = 0$  is optimal for B will increase. Otherwise it decreases. Only the actual derivatives are given below; full derivations can be obtained from the authors.

**Proof of Lemma 3:** The form of (17) directly implies the first part of the lemma: since  $c$  is positive,  $(2p - 1) > 0$  if the “skimming” region is to have a solution. The same is true for the “transition” and the “honest” regions by definition. The second part is derived similarly, using the simplified version of (6):

$$\frac{dx^*}{dw_0} = \frac{1 - 2p + \delta_M p^2}{(1 - \delta_M p)(1 - 2p)}. \quad (18)$$

With  $p > \frac{1}{2}$ , the denominator of (17) is negative, so  $\frac{dx^*}{dw_0} > 0$  whenever  $1 - 2p + \delta_M p^2 < 0$ , and less than or equal to zero otherwise. QED.

**Proof of Proposition 2:** For  $c$ , the total derivatives are:

$$\frac{dw_0^*}{dc} = \left[ \frac{(1 - \delta_M p)^3 (2p - 1)^3}{\delta_M^2 \beta^2 p^3 (1 - p)^3} \right] \frac{1}{Q}, \quad (19)$$

$$\frac{dx^*}{dc} = \left[ \frac{(1 - \delta_M p)^2 (2p - 1)^2}{\delta_M^2 \beta^2 p^3 (1 - p)^3} \right] \frac{-1 + 2p - \delta_M p^2}{Q}, \quad (20)$$

where  $Q = -2 + (6 + \delta_M)p - 3(2 + \delta_M)p^2 + 4\delta_M p^3$ , which is always negative for  $p > \frac{1}{2}$ . We will use  $Q$  throughout. (20) is of indeterminate sign. (19) is negative for  $p > \frac{1}{2}$ , as is the difference between (19) and (20), implying that  $p$  is decreasing in  $c$  as well in the “skimming” region. In the “transition” region, the equilibrium value of  $p$  is independent of  $c$  from (16). In the “honest” region, differentiating (9) yields  $\frac{dp_0}{dc} = -\frac{1}{2\beta} \left( \frac{1}{4} - \frac{c}{\beta} \right)^{-\frac{1}{2}} < 0$ . As  $p$  is unchanging in  $c$  in the transition region, and decreasing in  $c$  in both other regions, both cutoffs are increasing in  $c$ . QED.

**Proof of Proposition 3:** The total derivatives for  $b$  are

$$\frac{dw_0^*}{db} = \frac{(1 - \gamma)}{\gamma b^2 \delta_M}, \quad (21)$$

$$\frac{dx^*}{db} = \frac{(1-\gamma)}{\gamma b^2 \delta_M}, \quad (22)$$

and those for  $\gamma$  are

$$\frac{dw_0^*}{d\gamma} = \frac{1}{\gamma^2 b \delta_M}, \quad (23)$$

$$\frac{dx^*}{d\gamma} = \frac{1}{\gamma^2 b \delta_M}. \quad (24)$$

All are positive for  $p > \frac{1}{2}$ . The differences of each pair are zero, so  $p$  does not change with  $b$  or  $\gamma$ . In the “transition” region, the derivative with respect to  $b$  is:  $\frac{dp(w_0)}{db} = \frac{-(1-\gamma)\beta p^2(1-p)^2}{\gamma b^2(-1+2p-\delta_M p^2)} < 0$ , since (18) is positive in this region, and with respect to  $\gamma$  is:  $\frac{dp(w_0)}{d\gamma} = \frac{-\beta p^2(1-p)^2}{\gamma^2 b(-1+2p-\delta_M p^2)} < 0$  for the same reason. In the “honest” region, the probability of success depends on neither  $b$  nor  $\gamma$ . As  $p$  is decreasing in the transition region and constant in the skimming and honest regions, both cutoffs are decreasing in  $b$ . QED.

**Proof of Proposition 4** The partial derivative of  $x^*$  with respect to  $\delta_M$  is

$$\frac{\partial x^*}{\partial \delta_M} = \frac{-p}{\beta(1-\delta_M p)(2p-1)} [1 + \beta x^*(1-p)].$$

This is negative for  $p > \frac{1}{2}$ , but neither of the following two total derivatives can be signed:

$$\frac{dw_0^*}{d\delta_M} = \left( \frac{1-\gamma}{\delta_M^2 b \gamma} \right) + \left( \frac{-1}{\delta_M^2 \beta p(1-p)} \right) \left[ \frac{-1+2p-(2-\delta_M)p^2}{Q} \right]. \quad (25)$$

$$\frac{dx^*}{d\delta_M} = \left( \frac{1-\gamma}{\delta_M^2 b \gamma} \right) + \left( \frac{-1}{\delta_M^2 \beta p(1-p)} \right) \left[ \frac{-1+(2+\delta_M)p-2(1+\delta_M)p^2+2\delta_M p^3}{Q} \right]. \quad (26)$$

In each case the second term is always negative, while the first is always positive. The difference of the two derivatives is positive, though, implying that  $p$  is increasing in  $\delta_M$  in the “skimming” region. In the “transition” region,  $\frac{dp(w_0)}{d\delta_M} = \frac{p^2(1-p)}{-1+2p-\delta_M p^2} > 0$  since (18) is positive in this region. In the “honest” region, the probability of success is independent of  $\delta_M$ .  $p$  is increasing in  $\delta_M$  in all regions but the honest one. Thus  $\gamma_0$  is increasing in  $\delta_M$ . To discern if the same is true for  $\gamma_1$ , one needs to compare the relative rates of increase in  $\delta_M$ . For all  $\delta_M > 1/2$ ,  $p(w_0)$  increases more rapidly than  $p$ , implying that in this range  $\gamma_1$  is also increasing. QED.

**Proof of Proposition 5:** The total derivatives for  $\alpha$  are:

$$\frac{dw_0^*}{d\alpha} = 1, \quad (27)$$

$$\frac{dx^*}{d\alpha} = 0. \quad (28)$$

Increasing  $w^*$  and  $\alpha$  by the same amount leaves  $p$  unchanged in the “skimming” region.  $\alpha$  has no effect as well in the “honest” and “transition” regions, and both cutoffs are unchanging in it. The total derivatives for  $\beta$  are:

$$\frac{dw_0^*}{d\beta} = \frac{1}{\beta^2} \ln \left( \frac{p}{1-p} \right) - \frac{(1 - \delta_M p)(-2 - \delta_M + 4\delta_M p)}{\beta^2 \delta_M Q}, \quad (29)$$

$$\frac{dx^*}{d\beta} = \frac{(1 - \delta_M p)(2 - \delta_M)}{\beta^2 \delta_M Q}. \quad (30)$$

Although the first is usually negative, it goes positive for very high ( $\sim .96$ )  $p$ . The second is negative for all  $p > \frac{1}{2}$ . Subtracting the second from the first yields a positive result, so increasing  $\beta$  increases the probability that an attack will be successful in the “skimming” region. This is also true in the both the “transition” and “honest” cases, since:  $\frac{dp(w_0)}{d\beta} = \frac{p(1-p)(1-\delta_M p)}{\beta(-1+2p-\delta_M p^2)} > 0$  and  $\frac{dp_0}{d\beta} = \frac{c}{2\beta^2} \left( \frac{1}{4} - \frac{c}{\beta} \right)^{-\frac{1}{2}} > 0$ . As  $p$  is increasing in  $\beta$  in all three regions, we must compare relative rates to discern the effect on both cutoffs.  $p(w_0)$  increases more rapidly than both  $p$  and  $p_0$ , implying that both  $\gamma_0$  and  $\gamma_1$  are increasing. QED.

## 2 References

Ashworth, Scott and Ethan Bueno de Mesquita. 2006. Monotone Comparative Statics for Models of Politics. *American Journal of Political Science* 50:214-231.